

# POSITIVITY IN THE COHOMOLOGY OF FLAG BUNDLES (AFTER GRAHAM)

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In [Gr], Graham proves that the structure constants of the equivariant cohomology ring of a flag variety are positive combinations of monomials in the roots:

**Theorem 1** ([Gr, Cor. 4.1]). *Let  $X = G/B$  be the flag variety for a complex semisimple group  $G$  with maximal torus  $T \subset B$ , and let  $\{\sigma_w \in H_T^* X \mid w \in W\}$  be the basis of ( $B$ -invariant) Schubert classes. Let  $\{\alpha_i\}$  be the simple roots which are negative on  $B$ . Then in the expansion*

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w,$$

*the coefficients  $c_{uv}^w$  are in  $\mathbb{Z}_{\geq 0}[\alpha]$ .*

Graham deduces this from a more general result about varieties with finitely many unipotent orbits, which is proved using induction and a calculation in the rank-one case.

The goal of this note is to give a short, geometric proof of Graham's positivity theorem, based on a transversality argument. Here I only discuss type  $A$ , but other types work as well. (For a type-uniform version, a change of language is needed: one should replace vector bundles with corresponding principal  $G$ -bundles.)

Throughout,  $Fl$  denotes the variety of (complete) flags in  $\mathbb{C}^n$ , and if  $V \rightarrow X$  is a vector bundle,  $\mathbf{Fl}(V) \rightarrow X$  is the bundle of flags in  $V$ .

Recall that for  $T' \cong (\mathbb{C}^*)^n$ , we have  $BT' = (\mathbb{P}^\infty)^{\times n}$  and  $H_{T'}^* Fl = H^*(ET' \times^{T'} Fl) = H^*\mathbf{Fl}(E')$ , where  $E'$  is the sum of the  $n$  tautological line bundles on  $BT'$ . The *effective* action on  $Fl$  is by  $T \cong (\mathbb{C}^*)^n/\mathbb{C}^*$ , and the classifying space for this torus is  $BT = (\mathbb{P}^\infty)^{\times n-1}$ . We will usually deal with the effective torus.

Let  $\mathbb{P} = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$  ( $n-1$  factors), with  $m \gg 0$ , and write  $H^*\mathbb{P} = \mathbb{Z}[\alpha_1, \dots, \alpha_{n-1}]$ . (We always assume that  $m$  is large enough so that there are no relations in the relevant degrees.) Let  $M_i = p_1^*(\mathcal{O}(-1))$  be the tautological bundle on the  $i$ th factor, and let  $\alpha_i = -c_1(M_i)$ . Note that the class of any effective cycle in  $H^*\mathbb{P}$  is a positive polynomial in the  $\alpha$ 's.

Let

$$L_i = M_1 \otimes \cdots \otimes M_{i-1}$$

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*Date:* September 30, 2007.

for  $1 \leq i \leq n$  (so  $L_1 = \mathcal{O}$  is the trivial line bundle), and let  $E_i = L_1 \oplus \cdots \oplus L_i$ . Thus we have a flag  $E_\bullet$  in  $E = E_n$ . Let  $\tilde{E}_\bullet$  be the opposite flag, with  $\tilde{E}_i = L_n \oplus \cdots \oplus L_{n+1-i}$ . In the flag bundle  $p : \mathbf{Fl}(E) \rightarrow \mathbb{P}$ , with universal quotient flags  $Q_\bullet$ , we have Schubert loci  $\Omega_w = \Omega_w(E_\bullet \rightarrow Q_\bullet)$ , defined by

$$(1) \quad \Omega_w = \{x \in \mathbf{Fl}(E) \mid \text{rk}(E_p \rightarrow Q_q) \leq \#(i \leq q \mid w(i) \leq p)\}.$$

Opposite Schubert loci  $\tilde{\Omega}_w = \Omega_w(\tilde{E}_\bullet \rightarrow Q_\bullet)$  are defined similarly. We also have ‘‘Schubert cell bundles’’  $\Omega_w^o$ : these are affine bundles over  $\mathbb{P}$  which are open in the corresponding loci  $\Omega_w$ , and are defined by replacing the inequality in (1) with an equality.

The classes  $[\Omega_w]$  form a basis for  $H^*\mathbf{Fl}(E)$  over  $H^*\mathbb{P}$ , as  $w$  ranges over  $S_n$ . Writing

$$[\Omega_u] \cdot [\Omega_v] = \sum_w c_{uv}^w [\Omega_w]$$

with  $c_{uv}^w \in H^*\mathbb{P}$ , our main result is the following:

**Proposition 2.** *The polynomials  $c_{uv}^w$  are positive, that is,  $c_{uv}^w \in \mathbb{Z}_{\geq 0}[\alpha_1, \dots, \alpha_{n-1}]$ .*

This implies Graham’s positivity theorem (in this context), since  $\mathbb{P}$  approximates  $BT$  for  $m$  sufficiently large, and  $\mathbf{Fl}(E)$  approximates  $ET \times^T Fl$ , with  $[\Omega_w]$  corresponding to the equivariant class  $\sigma_w$ . (See [Fu2, §9].)

Proposition 2 is a consequence of a transversality statement:

**Proposition 3.** *For any  $u, v, w \in S_n$ , there is a translate  $\Omega'_v$  of  $\Omega_v$  by the action of a connected algebraic group such that  $\Omega'_v$  intersects  $\Omega_u$  and  $\tilde{\Omega}_{w_0 w}$  properly and generically transversally.*

To deduce Proposition 2, first note that the intersection  $\Omega_u \cap \tilde{\Omega}_{w_0 w}$  is always proper and generically transverse. Thus Proposition 3 says that  $\Omega'_v \cap (\Omega_u \cap \tilde{\Omega}_{w_0 w})$  is proper and generically transverse. By [Fu1, Ex. (8.1.11)], this says that

$$[\Omega_v] \cdot [\Omega_u] \cdot [\tilde{\Omega}_{w_0 w}] = [\Omega'_v \cap \Omega_u \cap \tilde{\Omega}_{w_0 w}].$$

(Since  $\Omega'_v = g \cdot \Omega_v$  for some  $g$  in a connected algebraic group,  $[\Omega'_v] = [\Omega_v]$ .) Using relative Poincaré duality (see e.g. [Fu2, §A.6]), we have

$$c_{uv}^w = p_*([\Omega_u] \cdot [\Omega_v] \cdot [\tilde{\Omega}_{w_0 w}]) = p_*([\Omega_u \cap \Omega'_v \cap \tilde{\Omega}_{w_0 w}]).$$

This is an effective class in  $H^*\mathbb{P}$ , so Proposition 2 follows.

*Proof of Proposition 3.* This is essentially an application of Kleiman’s theorem. The endomorphism bundle

$$\begin{aligned} \mathbf{End}(E) &= \bigoplus_{i,j} L_i^{-1} \otimes L_j \\ &= \left( \bigoplus_{i < j} M_i \otimes \cdots \otimes M_{j-1} \right) \oplus \mathcal{O}^{\oplus n} \oplus \left( \bigoplus_{i > j} M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1} \right) \end{aligned}$$

has global sections in lower-triangular matrices, so the group  $B$  of (invertible) lower-triangular matrices acts on  $\mathbf{Fl}(E)$ , fixing the flag  $\tilde{E}_\bullet$  and stabilizing  $\tilde{\Omega}_{w_0 w}$ . (Note that the entries of a matrix in  $B$  are global sections of the line bundles  $M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1}$ , i.e., multi-homogeneous polynomials. This is a connected group over  $\mathbb{C}$ , acting on a fiber  $p^{-1}(x) \subset \mathbf{Fl}(E)$  by first evaluating the sections at  $x$ .)

Now let  $H = (GL_{m+1})^{\times(n-1)}$ , and for  $b \in B$ , let  $b_x$  be the evaluation at  $x \in \mathbb{P}$  (so the action of  $b$  on  $p^{-1}(x)$  is by  $b_x$ ). Consider the semidirect product  $\Gamma = B \rtimes H$ , given by  $(h \cdot b \cdot h^{-1})_x = b_{h^{-1} \cdot x}$ . (This action of  $H$  on  $B$  is just the usual action of  $H$  on global sections of the equivariant vector bundle  $\mathbf{End}(E)$ .)<sup>1</sup> As a semidirect product of connected groups,  $\Gamma$  is a connected algebraic group. We claim that the locus  $\tilde{\Omega}_{w_0 w}^o$  is homogeneous for the action of  $\Gamma$ . Indeed,  $B$  acts transitively on each fiber of  $\tilde{\Omega}_{w_0 w}^o$ , and the action of  $H$  on  $\mathbf{Fl}(E)$  induces a transitive action on the set of fibers of  $\tilde{\Omega}_{w_0 w}^o$ . (The line bundles  $L_i$  are equivariant for  $H$ , so  $H$  preserves the flag  $\tilde{E}_\bullet$ , and therefore acts on  $\tilde{\Omega}_{w_0 w}$ .)

Finally, note that  $\Omega_u^o$  and  $\tilde{\Omega}_{w_0 w}^o$  intersect transversally, as do  $\Omega_v^o$  and  $\tilde{\Omega}_{w_0 w}^o$ . The proposition follows from Lemma 4 below, taking  $U = \Omega_u$ ,  $V = \Omega_v$ , and  $W = \tilde{\Omega}_{w_0 w}$ , with their stratifications by Schubert loci.  $\square$

**Lemma 4.** *Let  $X$  be a nonsingular variety over a field of characteristic 0, with an action of a connected algebraic group  $\Gamma$ . Let  $U, V, W \subset X$  be subvarieties with stratifications*

$$\begin{aligned} U_0 &\subset \cdots \subset U_\ell = U, \\ V_0 &\subset \cdots \subset V_m = V, \\ W_0 &\subset \cdots \subset W_n = W, \end{aligned}$$

*with each stratum  $U_i \setminus U_{i-1}$  nonsingular. Assume also that  $\Gamma$  acts on  $W$ , with each stratum  $W_i \setminus W_{i-1}$  a disjoint union of homogeneous spaces.*

*If  $U_i \setminus U_{i-1}$  meets  $W_k \setminus W_{k-1}$  transversally for all  $i, k$ , and similarly for  $V_j \setminus V_{j-1}$  and  $W_k \setminus W_{k-1}$ , then there is an element  $g \in \Gamma$  such that  $g \cdot V$  meets  $U \cap W$  properly and generically transversally.*

This can be deduced from results found in [Sp]; see also [Si] for a vast generalization. The proof of this version is quite short, so we give it here.

*Proof.* Applying Kleiman's theorem (cf. [Ha, III.10.8]) to the pairs  $(U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1})$  and  $(V_j \setminus V_{j-1} \cap W_k \setminus W_{k-1})$  inside the homogeneous space  $W_k \setminus W_{k-1}$ , we can choose  $g \in \Gamma$  such that each intersection

$$\begin{aligned} &(U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1}) \cap g \cdot (V_j \setminus V_{j-1} \cap W_k \setminus W_{k-1}) \\ &= (U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1}) \cap (g \cdot V_j \setminus g \cdot V_{j-1} \cap W_k \setminus W_{k-1}) \end{aligned}$$

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<sup>1</sup>Alternatively, one could take  $\Gamma$  to be the subgroup of  $\text{Aut}(\mathbf{Fl}(E))$  generated by the images of  $B$  and  $G$  via the homomorphisms corresponding to their respective actions.

is transverse, so the intersection  $U \cap W \cap g \cdot V$  is proper and generically transverse.  $\square$

**Remark 5.** All that is required in the proof of Proposition 3 are the facts that  $\mathbb{P}$  is homogeneous for the action of an algebraic group  $H$ , and  $L_i$  are  $H$ -equivariant line bundles such that  $L_i^{-1} \otimes L_j$  is globally generated for  $i > j$ .

**Remark 6.** To recover the result that for (type  $A$ ) equivariant Schubert calculus, the structure constants  $c_{uv}^w$  are in  $\mathbb{Z}_{\geq 0}[t_2 - t_1, \dots, t_n - t_{n-1}]$ , let  $\mathbb{P}' = (\mathbb{P}^{m'})^{\times n}$  and choose a map  $\varphi : \mathbb{P}' \rightarrow \mathbb{P}$  such that  $\varphi^* M_i = M'_i \otimes (M'_{i+1})^{-1}$ , where  $M'_i$  is the tautological bundle on the  $i$ th factor of  $\mathbb{P}'$ , with  $t_i = c_1(M'_i)$ . (Note that  $\varphi$  will not be holomorphic!)

The  $T'$ -equivariant class of a Schubert variety (for  $T' = (\mathbb{C}^*)^n$ ) can be identified with the class of the locus  $\Omega_w(E'_\bullet \rightarrow Q_\bullet) \subset \mathbf{Fl}(E')$ , where  $E'_i = M'_1 \oplus \dots \oplus M'_i$  is a flag of bundles on  $\mathbb{P}'$ . Since this is  $\varphi^{-1} \Omega_w$ , the equivariant structure constants are  $\varphi^* c_{uv}^w$ , which are positive in the variables  $\varphi^* \alpha_i = t_{i+1} - t_i$ .

**Remark 7.** The naive choice of flag, with  $F_i = M_1 \oplus \dots \oplus M_i$ , does not work: The bundle  $\mathbf{End}(F)$  has only diagonal global sections, so the corresponding loci  $\Omega_w^o$  are not homogeneous. This explains why one does not see positivity over  $\mathbb{P}'$ .

*Acknowledgements.* This proof was inspired by William Fulton's lectures on equivariant cohomology [Fu2], and I thank him for comments on the manuscript. Thanks also to Sue Sierra for interesting discussions, and for bringing [Sp] to my attention.

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